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Properties of Nonlocal Charges in the Supersymmetric Two Boson Hierarchy

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Abstract

We obtain the conserved, nonlocal charges for the supersymmetric two boson hierarchy from fractional powers of its Lax operator. We show that these charges reduce to the ones of the supersymmetric KdV system under appropriate reduction. We study the algebra of the nonlocal, local and supersymmetry charges with respect to the first and the second Hamiltonian structures of the system and discuss how they close as a graded nonlinear cubic algebra.

1. Introduction

Integrable models have been studied extensively in the past [1-4]. The supersymmetric generalizations of these models have also raised a lot of interest after the introduction of the supersymmetric KdV equation (sKdV) [5,6] and the supersymmetric nonlinear Schrödinger equations (sNLS) [7,8]. In a recent paper [9] we have constructed the supersymmetric version of the Two Boson equation (sTB) [10-13] also known as the dispersive long water wave equation [14-16]. This integrable system (the bosonic as well the supersymmetric version) has a very rich structure since it is tri-Hamiltonian, has a nonstandard Lax representation and reduces to well known integrable systems under appropriate reductions.

An interesting property of the supersymmetric integrable models is the existence of nonlocal conservation laws. These conserved charges were first obtained in [17] for the sKdV equation and in [7] for the sNLS equation through group symmetry analysis of these equations. In [18] a nice interpretation for the existence of the nonlocal charges for the sKdV was given: they can be obtained as the Adler supertrace of odd powers of the fourth root of the Lax operator.

In this paper we show that nonlocal conserved charges also arise in the sTB hierarchy and as the supertrace of odd powers of the square root of the Lax operator of our nonstandard system. The explicit form of the Hamiltonian structures of our system allows us to calculate the algebra of these nonlocal charges, the local ones and the supersymmetry charge. We find the algebra to be a nonlinear algebra with a cubic term much like the algebras of the nonlocal charges of the Heisenberg spin chain and the nonlinear sigma model [19-21]. This result indicates the presence of some sort of a Yangian structure in the supersymmetric integrable models [22], a result which deserves further investigation.

Our paper is organized as follows. In sec. 2 we construct the first four local conserved charges of the sTB equation (as well as the hierarchy) and review its tri-Hamiltonian structure. Here we also present the correct second and third Hamiltonian structures of the system which are slightly different from the ones given in [9]. In sec. 3 we construct the first three nonlocal conserved charges of our model and discuss various properties of these charges. In sec. 4 we calculate the algebra of the local, nonlocal and supersymmetry charges, using both the first and the second Hamiltonian structures and show that they

satisfy a graded cubic algebra. We present our conclusions as well as a short discussion on the algebra in sec. 5.

2. Local Charges for the Supersymmetric Two Boson Hierarchy

The supersymmetric Two Boson equation [9] is an integrable system represented by a Lax operator of the form

$$L = D^2 - (D\Phi_0) + D^{-1}\Phi_1 \tag{1}$$

and a nonstandard Lax equation

$$\frac{\partial L}{\partial t} = \left[L, (L^2)_{\geq 1}\right] \tag{2}$$

which leads to the equations

$$\begin{aligned}\frac{\partial \Phi_0}{\partial t} &= -(D^4\Phi_0) + (D(D\Phi_0)^2) + 2(D^2\Phi_1) \\ \frac{\partial \Phi_1}{\partial t} &= (D^4\Phi_1) + 2D^2((D\Phi_0)\Phi_1)\end{aligned} \tag{3}$$

Here

$$\begin{aligned}\Phi_0 &= \psi_0 + \theta J_0 \\ \Phi_1 &= \psi_1 + \theta J_1\end{aligned} \tag{4}$$

are two fermionic superfields, the covariant derivative in the superspace has the form

$$D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial x} \tag{5}$$

and the canonical dimensions of various quantities are given by

$$\begin{aligned}[x] &= -1 & [\Phi_0] &= \frac{1}{2} \\ [t] &= -2 & [\Phi_1] &= \frac{3}{2} \\ [\theta] &= -\frac{1}{2} & &\end{aligned} \tag{6}$$

The local conserved charges can be obtained from

$$Q_n = \mathrm{sTr}\, L^n = \int dz\, \mathrm{sRes}\, L^n \qquad n = 1, 2, \ldots \tag{7}$$

where "sRes" stands for the super residue which is defined to be the coefficient of the D^{-1} term in the pseudo super-differential operator with D^{-1} at the right ($D^{-1} = \partial^{-1}D$) and $z \equiv (x, \theta)$ is the superspace coordinate. The first ones have the explicit form

$$
\begin{aligned}
Q_1 =& -\int dz\, \Phi_1 \\
Q_2 =& 2\int dz\, (D\Phi_0)\Phi_1 \\
Q_3 =& 3\int dz\, \Big[(D^3\Phi_0) - (D\Phi_1) - (D\Phi_0)^2\Big]\Phi_1 \\
Q_4 =& 2\int dz\, \Big[2(D^5\Phi_0) + 2(D\Phi_0)^3 + 6(D\Phi_0)(D\Phi_1) - 3\left(D^2(D\Phi_0)^2\right)\Big]\Phi_1
\end{aligned}
\tag{8}
$$

Note that $[Q_n] = n$, they are bosonic and manifestly supersymmetric since supersymmetry can be thought of as translations of the Grassmann coordinate in the superspace (See also eq. (32).).

Defining the Hamiltonians as

$$
H_n = \frac{(-1)^{n+1}}{n} Q_n \tag{9}
$$

we note that the system, eq.(3), has a tri-Hamiltonian structure of the form

$$
\partial_t \begin{pmatrix} \Phi_0 \\ \\ \Phi_1 \end{pmatrix} = \mathcal{D}_1 \begin{pmatrix} \frac{\delta H_3}{\delta \Phi_0} \\ \\ \frac{\delta H_3}{\delta \Phi_1} \end{pmatrix} = \mathcal{D}_2 \begin{pmatrix} \frac{\delta H_2}{\delta \Phi_0} \\ \\ \frac{\delta H_2}{\delta \Phi_1} \end{pmatrix} = \mathcal{D}_3 \begin{pmatrix} \frac{\delta H_1}{\delta \Phi_0} \\ \\ \frac{\delta H_1}{\delta \Phi_1} \end{pmatrix} \tag{10}
$$

where

$$
\mathcal{D}_1 = \begin{pmatrix} 0 & -D \\ -D & 0 \end{pmatrix} \tag{11}
$$

which yields the following nonvanishing Poisson brackets in components (We list them for our later use in the calculation of the charge algebra in section 4.)

$$
\begin{aligned}
\{\psi_0(x), \psi_1(y)\}_1 =& -\delta(x-y) \\
\{J_0(x), J_1(y)\}_1 =& \delta'(x-y)
\end{aligned}
\tag{12}
$$

The second structure is given by

$$
\mathcal{D}_2 = \begin{pmatrix} -2D - 2D^{-1}\Phi_1 D^{-1} + D^{-1}(D^2\Phi_0)D^{-1} & D^3 - D(D\Phi_0) + D^{-1}\Phi_1 D \\ \\ -D^3 - (D\Phi_0)D - D\Phi_1 D^{-1} & -\Phi_1 D^2 - D^2\Phi_1 \end{pmatrix} \tag{13}
$$

which gives the nonvanishing Poisson brackets in the components of the form

$$
\begin{aligned}
\{\psi_0(x),\psi_0(y)\}_2 =&\Big(\partial^{-1}J_0'\big(\partial^{-1}\delta(x-y)\big)\Big)-2\Big(\partial^{-1}J_1\big(\partial^{-1}\delta(x-y)\big)\Big)-2\delta(x-y)\\
\{\psi_0(x),J_0(y)\}_2 =&\Big(\partial^{-1}\psi_0'\delta(x-y)\Big)-2\Big(\partial^{-1}\psi_1\delta(x-y)\Big)\\
\{J_0(x),J_0(y)\}_2 =&2\delta'(x-y)\\
\{\psi_0(x),\psi_1(y)\}_2 =&\Big(\partial^{-1}J_1\delta(x-y)\Big)-J_0\delta(x-y)+\delta'(x-y)\\
\{\psi_0(x),J_1(y)\}_2 =&\psi_0'\delta(x-y)+\Big(\partial^{-1}\psi_1\delta'(x-y)\Big)\\
\{J_0(x),\psi_1(y)\}_2 =&\psi_1\delta(x-y)\\
\{J_0(x),J_1(y)\}_2 =&(J_0\delta(x-y))'-\delta''(x-y)\\
\{\psi_1(x),J_1(y)\}_2 =&2\psi_1\delta'(x-y)+\psi_1'\delta(x-y)\\
\{J_1(x),J_1(y)\}_2 =&J_1'\delta(x-y)+2J_1\delta'(x-y)
\end{aligned}
\tag{14}
$$

In (14) as well as in (12) we remember that $\{A,B\}=-(-1)^{|A||B|}\{B,A\}$ and the parenthesis limit the action of the inverse (integral) operators. We would use this convention through out the paper. Introducing the recursion operator

$$
R=\mathcal{D}_2\mathcal{D}_1^{-1} \tag{15}
$$

the third Hamiltonian structure can be written as (One can write it out explicitly, but its form is not very illuminating.)

$$
\mathcal{D}_3=R\,\mathcal{D}_2 \tag{16}
$$

Finally, through the recursion operator defined in (15) we can relate the local conserved charges in a recursive manner as

$$
\begin{pmatrix} \frac{\delta H_{n+1}}{\delta\Phi_0} \\ \\ \frac{\delta H_{n+1}}{\delta\Phi_1} \end{pmatrix} = R^\dagger \begin{pmatrix} \frac{\delta H_n}{\delta\Phi_0} \\ \\ \frac{\delta H_n}{\delta\Phi_1} \end{pmatrix} \tag{17}
$$

where

$$
R^\dagger=\begin{pmatrix} D^2-D^{-1}(D^2\Phi_0)+(D\Phi_0)+\Phi_1D^{-1} & 2(D\Phi_1)-2\Phi_1D-D^{-1}(D^2\Phi_1) \\ \\ 2+D^{-2}\Phi_1D^{-1}-D^{-2}(D^2\Phi_0)D^{-1} & -D^2-D^{-2}(D\Phi_1)+(D\Phi_0)+D^{-1}\Phi_1 \end{pmatrix} \tag{18}
$$

Here we note that the second Hamiltonian structure in equation (13) differs slightly from $\widetilde{\mathcal{D}}_2$ given in [9]. Even though $\widetilde{\mathcal{D}}_2$ also gives the right equations when used in (10) and defines a recursion operator (as in (15)) which relates the lower order conserved charges (as in (17)), it differs from $\mathcal{D}_2$ by the presence of nonlocal field dependent terms in the latter. The difference in the two structures, therefore, gives no contribution to the Hamiltonian equations (as can be explicitly checked) and yet is crucial for Jacobi identity to hold. In fact, using the prolongation methods described in [23] (and generalized to the supersymmetric systems in [24]), the bivector associated with $\widetilde{\mathcal{D}}_2$

$$\Theta_{\widetilde{\mathcal{D}}_2} = \frac{1}{2}\sum_{\alpha,\beta}\int dz\,\left((\widetilde{\mathcal{D}}_2)_{\alpha\beta}\Omega_\beta\right)\wedge\Omega_\alpha \qquad \alpha,\beta=0,1 \tag{19}$$

yields a field independent term under prolongation

$$\mathbf{pr}\,\vec{v}_{\widetilde{\mathcal{D}}_2\vec{\Omega}}(\Theta_{\widetilde{\mathcal{D}}_2}) = -\int dz\,(D\Omega_0)\wedge(D\Omega_0)\wedge(D\Omega_1)\neq 0 \tag{20}$$

which implies that $\widetilde{\mathcal{D}}_2$ does not satisfy the Jacobi identity. However, it can be checked, in a straightforward but tedious manner that

$$\mathbf{pr}\,\vec{v}_{\mathcal{D}_2\vec{\Omega}}(\Theta_{\mathcal{D}_2}) = 0 \tag{21}$$

Therefore, $\mathcal{D}_2$ satisfies the Jacobi identity and represents a true Hamiltonian operator. Similar results also hold for $\mathcal{D}_3$. Thus, $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ represent the true Hamiltonian structures of this theory.

3. Nonlocal Charges for the Supersymmetric Two Boson Hierarchy

Existence of nonlocal conserved charges in supersymmetric integrable models such as the sKdV equation were first recognized in [17]. However, in ref. [18], a systematic procedure for their construction was given within the framework of the Gelfand-Dikii formalism. The authors in [18] realized that while the local charges for the sKdV can be obtained from odd powers of the square root of the Lax operator, $L^{\frac{2n-1}{2}}$, the nonlocal ones can be obtained from odd powers of the quartic roots, $L^{\frac{2n-1}{4}}$. For the sTB, the situation is quite similar. Since the local charges Q_n's are given by integer powers of L (Eq. (7)),

we expect to generate conserved nonlocal charges F_n's from odd powers of the square root of L, that is,

$$F_{\frac{2n-1}{2}} = \mathrm{sTr}\, L^{\frac{2n-1}{2}} \qquad n = 1, 2, \ldots \tag{22}$$

The square root of L given in (1) can be written as

$$L^{1/2} = D + a_0 + a_1 D^{-1} + a_2 D^{-2} + a_3 D^{-3} + a_4 D^{-4} + a_5 D^{-5} + \cdots \tag{23}$$

where

$$\begin{aligned}
a_0 =& 2(D^{-2}\Phi_1) - \Phi_0 \\
a_1 =& -(D^{-1}\Phi_1) \\
a_2 =& (D^{-1}(\Phi_0\Phi_1)) - 2(D^{-2}((D\Phi_0)\Phi_1)) + \Phi_0(D^{-1}\Phi_1) - \Phi_1 \\
a_3 =& \frac{1}{2}(D^{-1}\Phi_1)^2 - (D\Phi_0)(D^{-1}\Phi_1) + (D^{-1}((D\Phi_0)\Phi_1)) + (D\Phi_1) \\
\int dz\, a_5 =& \int dz \left[\mathcal{O}(Da_1) - \Big(D^{-1}\big(2a_3(D^2 a_0) - a_1 a_3 + (Da_1)(D\mathcal{O})\big)\Big)\right]
\end{aligned} \tag{24}$$

and we have defined (We note here that for the first three nonlocal charges that we will list below, we do not need a_4 and need only the integrated form of a_5.)

$$\mathcal{O} \equiv (D^2 a_0) + (Da_1) - 2a_2 \tag{25}$$

The grading of the coefficients are

$$|a_n| = n + 1 \tag{26}$$

In what follows, the relation

$$(-1)^{|A|}\big(D^{-1}(AB)\big) = A(D^{-1}B) - \Big(D^{-1}\big((DA)(D^{-1}B)\big)\Big) \tag{27}$$

is very useful and can be easily proved through the Leibnitz rule. Also, to perform integration by parts we need the generalized formula [18] which holds for local functions A and B,

$$\int dz\, (D^n A)B = (-1)^{n|A| + \frac{n(n+1)}{2}} \int dz\, A(D^n B), \quad \text{for all } n \tag{28}$$

We note, however, that for nonlocal functions, the surface terms can not always be neglected.

The first three nonlocal charges can be obtained after some long, but straightforward, calculations

$$\begin{aligned}
F_{1/2} &= -\int dz\,(D^{-1}\Phi_1)\\
F_{3/2} &= -\int dz\left[\frac{3}{2}(D^{-1}\Phi_1)^2 - \Phi_0\Phi_1 - \Big(D^{-1}\big((D\Phi_0)\Phi_1\big)\Big)\right]\\
F_{5/2} &= -\int dz\left[\frac{1}{6}(D^{-1}\Phi_1)^3 - \big(5(D^{-2}\Phi_1)\Phi_1 - 2\Phi_0\Phi_1 - 3(D\Phi_1) - (D^{-1}\Phi_1)^2\big)(D\Phi_0)\right.\\
&\qquad\left. + \Big(D^{-1}\big((D\Phi_1)\Phi_1 + \Phi_1(D\Phi_0)^2 - (D\Phi_1)(D^2\Phi_0)\big)\Big)\right]
\end{aligned} \tag{29}$$

These charges can be explicitly checked to be conserved under the flow (3). In fact, it can be shown from the structure of the Lax equation (2) that the nonlocal charges defined in (22) are indeed conserved under the flow of the sTB hierarchy.

It is worth noting here some of the interesting properties of these charges before evaluating their algebra. First of all, all the nonlocal charges are fermionic and $[F_{\frac{2n-1}{2}}] = \frac{2n-1}{2}$. Second, even though these charges are expressed as superspace integrals, they are not invariant under the supersymmetry transformations of the system (See also eq. (34).). This is mainly because of the nonlocality in the integrands. (We emphasize here that a superspace integral of a local function of superfields is automatically supersymmetric – not necessarily true for nonlocal functions.) However, we also note that there is nothing wrong with these charges not being supersymmetric. After all, the supersymmetry charge, in these integrable models, is not supersymmetric. Rather it satisfies the graded Lie algebra

$$\{Q, Q\} = P \tag{30}$$

where P denotes the momentum operator. As we will show, these nonlocal charges are also part of an interesting graded algebra.

We also note that the nonlocal charges of the sTB hierarchy in (29) reduce to those of the sKdV hierarchy, up to normalizations [18], when we set $\Phi_0 = 0$. This is not surprising since we have already shown earlier [9] that the sKdV system is contained in the sTB system. However, unlike the sKdV system, here the nonlocal charges are not related recursively by either R or $R^\dagger$ of eq. (18). In fact, we have tried to construct systematically

an operator which will relate the nonlocal charges recursively, but have not succeeded. We can only speculate that these fermionic charges presumably generate fermionic flows with distinct Hamiltonian structures of their own which in turn can give a “recursion” operator connecting them. However, we would also like to add that these fermionic flows cannot have a Lax representation of the form $\frac{\partial L}{\partial \beta} = [L, (L^{1/2})_{\geq 1}]$, with β as the odd time, since such an equation is not consistent. In [25] odd flows based on Jacobian supersymmetric KP-hierarchies were studied for the sKdV case. This may give some insight to our present system if we use, instead, a nonstandard supersymmetric KP-hierarchy [26]. We do not have anything to add to this at this time. Finally, using the transformation given in [9] it is possible to obtain the nonlocal charges for the sNLS equation which should coincide with the ones obtained in [7].

4. Algebra of the Charges

First, let us note that the sTB equation is invariant under supersymmetry transformations generated by the following conserved fermionic charge (As we have noted earlier, the supersymmetry charge is not supersymmetric and, consequently, cannot be written as an integral of a local function in superspace. Therefore, it is much more convenient to work in components.)

$$Q = -\int dx \; (\psi_1 J_0 + \psi_0 J_1) \tag{31}$$

This charge together with the first Hamiltonian structure in (12), generates the supersymmetry transformations

$$\begin{aligned} \delta J_0 =& \epsilon \{Q, J_0\}_1 = \epsilon \psi_0' \\ \delta J_1 =& \epsilon \{Q, J_1\}_1 = \epsilon \psi_1' \\ \delta \psi_0 =& \epsilon \{Q, \psi_0\}_1 = \epsilon J_0 \\ \delta \psi_1 =& \epsilon \{Q, \psi_1\}_1 = \epsilon J_1 \end{aligned} \tag{32}$$

where ϵ is a constant Grassmann parameter of transformation.

It can now be easily shown with eqs. (31) and (12) that

$$\{Q, Q\}_1 = -Q_2 \tag{33}$$

which implies that the supersymmetry charge is not supersymmetric – rather it satisfies a graded Lie algebra. (We note here that, for conventional Lie algebra symmetries, the change in any variable A under a symmetry transformation generated by a charge G is given, except for factors, as $\delta A = \{G, A\}$. The above result, therefore, would say that under a supersymmetry transformation $\delta Q \neq 0$.).

As we have pointed out earlier even the nonlocal charges are not invariant under supersymmetry transformations and, in fact, we can easily calculate and see that

$$\begin{aligned}
\{Q, F_{1/2}\}_1 =& Q_1 \\
\{Q, F_{3/2}\}_1 =& \frac{1}{2} Q_2 \\
\{Q, F_{5/2}\}_1 =& \frac{1}{3} Q_3 + \frac{1}{24} Q_1^3
\end{aligned} \tag{34}$$

We also note that since the supersymmetry charge Q as well as the nonlocal charges $F_{\frac{2n-1}{2}}$ are conserved, they are in involution with the local charges Q_n (This can be explicitly checked for the flow in (3).), i.e,

$$\begin{aligned}
\{Q_n, Q_m\}_1 =& 0 \\
\{Q_n, F_{\frac{2m-1}{2}}\}_1 =& 0 \\
\{Q_n, Q\}_1 =& 0
\end{aligned} \tag{35}$$

The algebra of the nonlocal charges is interesting as well. We list here only the first few relations

$$\begin{aligned}
\{F_{1/2}, F_{1/2}\}_1 =& 0 \\
\{F_{1/2}, F_{3/2}\}_1 =& Q_1 \\
\{F_{1/2}, F_{5/2}\}_1 =& Q_2 \\
\{F_{3/2}, F_{3/2}\}_1 =& 2Q_2 \\
\{F_{3/2}, F_{5/2}\}_1 =& \frac{7}{3} Q_3 + \frac{7}{24} Q_1^3 \\
\{F_{5/2}, F_{5/2}\}_1 =& 3Q_4 - \frac{5}{8} Q_1^2 Q_2
\end{aligned} \tag{36}$$

This shows that the algebra of the conserved charges Q, Q_n and $F_{\frac{2n-1}{2}}$, at least at these orders, closes with respect to the first Hamiltonian structure in (12). However, the algebra is not a linear Lie algebra. Rather it is a graded nonlinear algebra where the nonlinearity

manifests in a cubic term. In fact, the canonical dimensions of the charges allows for higher order nonlinearity to be present even in the algebraic relations of these lower order charges, but the algebra appears to involve only a cubic nonlinearity. The Jacobi identity is seen to be trivially satisfied for this algebra since the Q_n's are in involution with all the fermionic charges (Q_n's are the Hamiltonians for the system and the fermionic charges are conserved.). We note here that the cubic terms in (34) and (36) arise from boundary contributions when nonlocal terms are involved. This can be seen simply using the following example. If we use the following realization of the inverse operator

$$\left(\partial^{-1}f(x)\right) = \frac{1}{2}\int dy\,\epsilon(x-y)f(y)\,, \qquad \epsilon(x) = \begin{cases} -1, & x<0 \\ 0, & x=0 \\ +1, & x>0 \end{cases} \tag{37}$$

then, this will lead, for instance, to boundary terms of the type

$$\int dz\,(D^{-1}\Phi_1)^2\Phi_1 = \frac{1}{3}\int dz\,D(D^{-1}\Phi_1)^3 = -\frac{1}{12}Q_1^3 \tag{38}$$

and this is the origin of the nonlinear terms. We would also like to point out that we have not succeeded in finding a closed form expression for a general element of the algebra in (34) and (36), but we will discuss in the conclusion why we believe the algebra to be a cubic algebra.

The sTB hierarchy has three distinct Hamiltonian structures. Consequently, one can ask other interesting questions such as what transformations does Q generate with the second structure or what is the algebra of the charges with respect to the second structure and so on. The supersymmetry transformations generated by Q with respect to the second structure can be easily calculated and have the form

$$\begin{aligned}
\delta J_0 =& \epsilon\{Q,J_0\}_2 = \epsilon\left(2\psi_1' + J_0\psi_1 + (\psi_0 J_0)' - \psi_0'' + (\partial^{-1}J_1)\psi_0' - 2(\partial^{-1}J_1)\psi_1\right)\\
\delta J_1 =& \epsilon\{Q,J_1\}_2 = \epsilon\left(\psi_1'' + 2(J_0\psi_1)' + (J_1\psi_0)' - (\psi_1(\partial^{-1}J_1))'\right)\\
\delta \psi_0 =& \epsilon\{Q,\psi_0\}_2 = \epsilon\left(J_0^2 - J_0' - J_0(\partial^{-1}J_1) + \psi_0\psi_0' + 2J_1 + 2(\partial^{-1}((\partial^{-1}J_1)J_1))\right)\\
\delta \psi_1 =& \epsilon\{Q,\psi_1\}_2 = \epsilon\left(2\psi_0'\psi_1 + \psi_0\psi_1' + J_1' + J_1(\partial^{-1}J_1) + J_1 J_0\right)
\end{aligned} \tag{39}$$

We note that these are highly nonlinear fermionic transformations and are nonlocal. We have explicitly verified that these transformations define a symmetry of the equations of motion of the supersymmetric Two Boson system.

The algebra of the charges can again be calculated with respect to the second Hamiltonian structure and we obtain

$$\{Q, Q\}_2 = \frac{2}{3}Q_3 - \frac{1}{6}Q_1^3 \tag{40}$$

This shows that, with respect to the second structure, Q satisfies a cubic graded algebra. Furthermore, with the nonlocal charges, it gives

$$\begin{aligned}\{Q, F_{1/2}\}_2 =& -\frac{1}{2}Q_2\\ \{Q, F_{3/2}\}_2 =& -\frac{1}{3}Q_3 + \frac{1}{12}Q_1^3\\ \{Q, F_{5/2}\}_2 =& -\frac{1}{4}Q_4 + \frac{1}{16}Q_1^2Q_2\end{aligned} \tag{41}$$

The charges Q and $F_{\frac{2n-1}{2}}$ are in involution with Q_n

$$\begin{aligned}\{Q_n, Q_m\}_2 =& 0\\ \{Q_n, F_{\frac{2m-1}{2}}\}_2 =& 0\\ \{Q_n, Q\}_2 =& 0\end{aligned} \tag{42}$$

and the algebra among the nonlocal charges has the form

$$\begin{aligned}\{F_{1/2}, F_{1/2}\}_2 =& 0\\ \{F_{1/2}, F_{3/2}\}_2 =& -\frac{1}{2}Q_2\\ \{F_{1/2}, F_{5/2}\}_2 =& -\frac{2}{3}Q_3 - \frac{1}{12}Q_1^3\end{aligned} \tag{43}$$

We see that the charges satisfy a graded algebra with respect to the second structure and the algebra continues to be a cubic algebra. In fact, we note that (41) and (43) represent a sort of shifting of (34) and (36). We believe that these general qualitative features would continue to hold even for the third Hamiltonian structure.

5. Conclusion

In this letter, we have constructed the conserved, nonlocal, fermionic charges for the sTB hierarchy. We have also presented the three, correct Hamiltonian structures of the system satisfying Jacobi identity which is checked through super-prolongation methods.

Our nonlocal charges are not related by the recursion operator of the theory even though they reduce to the charges of the sKdV system under appropriate reduction. The fermionic, nonlocal charges are not supersymmetric. Rather, the bosonic and the fermionic charges of the system Q_n, Q and $F_{\frac{2n-1}{2}}$ satisfy a graded algebra which is nonlinear and appears to be cubic. This structure of our algebra continues to hold even when evaluated with the second Hamiltonian structure of the system and we believe the third structure also will lead to similar conclusion.

We would like to point out here that the appearance of cubic terms in the algebra of nonlocal charges in this integrable model is nothing new. When carefully evaluated, such terms are also present in the algebra of nonlocal charges in the case of the sKdV system [18] even though it has not been observed before. For example, if we take the sKdV equation, $\Phi_t = -(D^6\Phi) + 3D^2(\Phi(D\Phi))$ and use the nonlocal charge

$$J_{3/2} = \int dz\, (D^{-1}\Phi)^2 \tag{44}$$

as well as the second Hamiltonian structure, as in [18] (The first Hamiltonian structure in the case of sKdV is not very convenient for calculations.), we obtain (See eq. (38).)

$$\{J_{3/2}, J_{3/2}\} = 4\int dz\, \left(\Phi(D\Phi) + \Phi(D^{-1}\Phi)^2\right) = 4H_3 - \frac{1}{3}H_1^3 \tag{45}$$

where we have defined the conserved local charges of sKdV as (the normalization is different from [18])

$$H_{2n-1} = \frac{2^{2n-1}}{2n-1}\operatorname{Tr} L^{\frac{2n-1}{2}} \tag{46}$$

Cubic algebras have been found earlier in studies of other systems such as the Heisenberg spin chains and the nonlinear sigma models [19-21] and appear to be a common feature when nonlocal charges are involved. In fact, on general grounds, one can argue that the nonlinearity in these algebras can be high. However, it is possible to redefine the generators in a highly nonlinear and nontrivial way such that the algebra is indeed cubic. This is quite well known in the case of the nonlinear sigma model [21]. Here we indicate how it can be done for the sKdV system which is simpler compared to the sTB hierarchy.

The following algebra can be derived for the sKdV in a straightfoward manner

$$
\begin{aligned}
\{J_{1/2}, J_{1/2}\} =& - H_1 \\
\{J_{1/2}, J_{3/2}\} =& 0 \\
\{J_{1/2}, J_{5/2}\} =& - 6H_3 - \frac{1}{4}H_1^3 \\
\{J_{3/2}, J_{3/2}\} =& 4H_3 - \frac{1}{3}H_1^3 \\
\{J_{3/2}, J_{5/2}\} =& 0 \\
\{J_{5/2}, J_{5/2}\} =& - 36H_5 - \frac{9}{80}H_1^5
\end{aligned}
\tag{47}
$$

Here the structure constants can be further simplified by rescaling $J_{\frac{2n-1}{2}}$, but we do not bother about it here. The important point to note is the appearance of the quintic term in the bracket of $J_{5/2}$ with itself. Let us note, however, that we can redefine

$$
\begin{aligned}
\hat{J}_{1/2} =& J_{1/2} \\
\hat{J}_{3/2} =& J_{3/2} \\
\hat{J}_{5/2} =& J_{5/2} + \alpha H_1^2 J_{1/2}
\end{aligned}
\tag{48}
$$

where α can be chosen such that the algebra becomes cubic. (Choice of α eliminates the quintic term. We note here that α turns out to be complex in this case. This may suggest a different normalization for the charges.) From this we strongly believe that one can redefine the charges even in the case of sTB such that the right hand side of the algebra in (34), (36), (41) and (43) will have the closed form structure

$$
a\,\hat{Q}_n + b \sum_{p+q+\ell=n} \hat{Q}_p \hat{Q}_q \hat{Q}_\ell \tag{49}
$$

where a and b are numerical factors and n is the sum of the canonical dimensions of the left hand side of the algebra.

We would conclude by noting here that the cubic algebras of this sort can be related with Yangians [19-22]. So, it is natural to expect that the algebra, in the present systems (both sKdV and sTB) also corresponds to a Yangian. There is, however, a difficulty with this. Namely, a Yangian starts out with a non-Abelian Lie algebra for the local charges. Here, in contrast, the algebra of the local charges, Q_n's, is involutive (Abelian). There

may still be an underlying Yangian structure in this algebra and this remains an open question.

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